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EFFICIENT FACETS AND RATES OF CHANGE: GEOMETRY AND ANALYSIS OF SOME PARETO-EFFICIENT EMPIRICAL PRODUCTION POSSIBILITY SETS

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Abstract

Efficient facets for the additive model of DEA (Data Envelopment Analysis) are obtained by a new series of linear programming models which are used to show how rates of change of outputs with respect to inputs, both analytically and computationally, may be determined along a given efficient facet. The differences between single and multiple output cases are determined and exemplified via a "cone direction" development.

Keywords: Data Envelopment Analysis, Efficiency Facet Tradeoffs, Production Efficiency Tradeoffs, and Tradeoff Geometry

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1. Introduction

A variety of models have now been developed within the methodology of Data Envelopment Analysis (DEA) for evaluating the (relative) efficiency of Decision Making Units (DMUs) which use multiple inputs to produce multiple outputs.

Building on the engineering ratio idea of a single-input, single-output efficiency measure, Charnes, Cooper and Rhodes (1978, 1981) generalized this notion to multiple-input, multiple-output situations and pushed forward on both their managerial aspects and, on the dual side of the mathematical programs involved, to more classical notions of Pareto efficiency or optimality. Subsequent extensions and elaborations in DEA include the "multiplicative" models of Charnes, Cooper, Seiford and Stutz (1982, 1983), the "modified ratio" model of Banker, Charnes and Cooper (1984), the "additive" model of Charnes, Cooper, Golany, Seiford and Stutz (1985), and the "extended additive" model of Charnes, Cooper, Rousseau and Semple (1987). These works provide alternative (but related) notions of efficiency with respect to different empirically-defined production possibility or reference sets. Section 2 and Figure 1 summarize the construction of these different reference sets.

The identification of these models, indeed the whole of DEA, as Charnes-Cooper (1961, ch. IX) tests for Pareto-efficiency of the generators of the empirically defined production possibility set was made by Charnes, Cooper and Seiford (1981) and elaborated in Charnes et al (1985). They also began investigation of (among other informatics and function properties) the efficient facets of empirical production possibility sets as a step toward determining the rates of change of outputs with changes in inputs or other substitutional rates along those facets. These rates of change have important economic and managerial implications in trade-off analysis and resource allocation.

In the present paper we extend and develop in depth the insights and beginnings in Charnes et al (1985). Section 3 provides a theoretical basis for identification of the efficient facets of the empirical production possibility set that underlies the additive model of DEA. The efficient facets may be obtained by solving a new series of linear programming problems, one for each DEA efficient observed input-output point. Section 4 then shows analytically and computationally how rates of change of outputs with respect to inputs can be determined along a given efficient facet. These rates of change are computed from a linearly independent (in the inputs) subset of the facet points. They will be different along different facets. In the single output case we always obtain nonnegative rates of change, but with multiple outputs this is not guaranteed. The "cone direction" development in Section 5, in either the ouput space or the input space, show us what combinations of associated substitutions are needed in order to obtain nonnegative rates of change. A simple two-output, two-input, ten-DMU example is carried throughout the paper to illustrate what is involved. Concluding remarks that are given in Section 6.
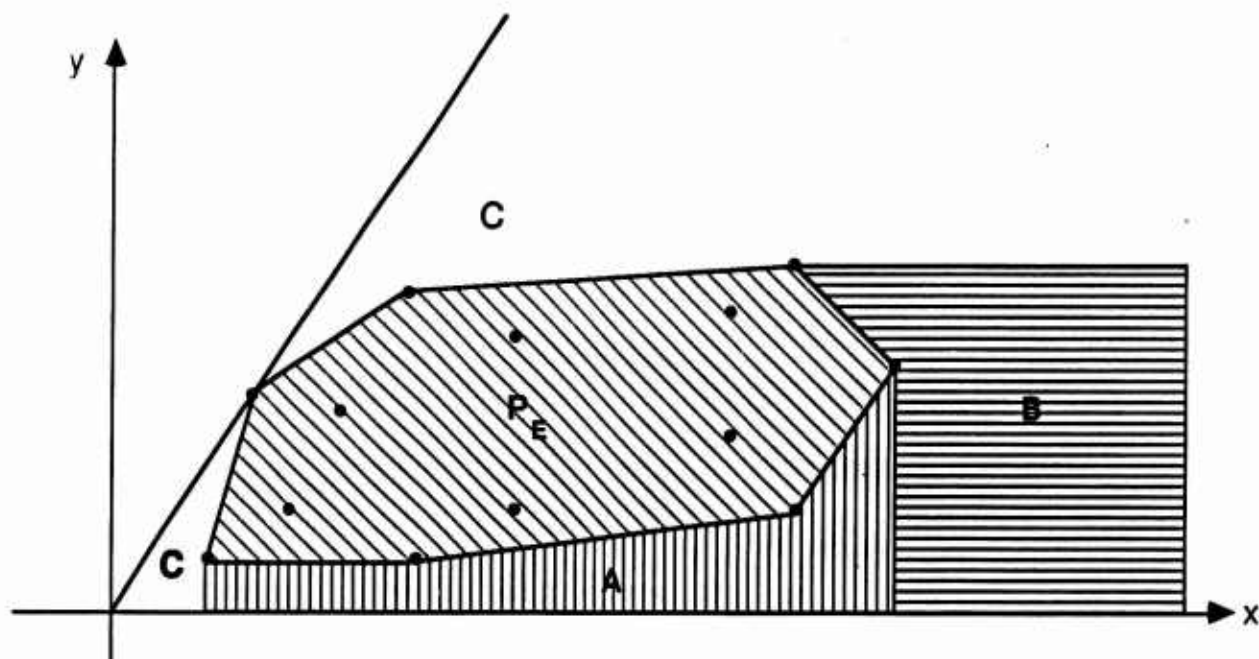
2. Empirical Production Possibility Sets

Consider the (empirical) points (x_j, y_j), $j = 1, \ldots n$, where the x_j are (mx1) input vectors and the y_j are (sx1) output vectors. In most applications they will be positive or nonnegative vectors. We define the 'empirical production set', P_E, to be the convex hull of these empirical points, that is,

$$P_E \equiv \{ (x, y): x = \sum_{j=1}^{n} x_j \mu_j, \; y = \sum_{j=1}^{n} y_j \mu_j, \; \forall \mu_j \geq 0, \; \sum_{j=1}^{n} \mu_j = 1 \} \qquad (2.1)$$

as shown in Figure 1.

Figure 1.

The 'empirical production possibility set' Q_E of Charnes et al (1985) is defined by adding to P_E all points with inputs in P_E and outputs not greater than some output in P_E that is,

$$Q_E \equiv \{ (x, y) : x = \overline{x}, \; y \leq \overline{y} \text{ for some } (\overline{x}, \overline{y}) \in P_E \} \qquad (2.2)$$

Thus $Q_E = P_E \cup A$ in Figure 1.

The Banker, Charnes and Cooper (1984) production possibility set adds to Q_E the set

$$\{ (x, y): x \geq \overline{x}, y = \overline{y} \quad \text{for some } (\overline{x}, \overline{y}) \in Q_E \}$$

is given by $Q_E \cup B$ in Figure 1.

The production possibility sets studied by Farrell (1957), Shephard (1970), and Färe and Lovell (1978) are truncated cones, given by $Q_E \cup B \cup C$ in Figure 1.

For efficient production we wish to maximize on outputs while minimizing on inputs. Thus we set

$$g_k (x, y) = \begin{cases} -y_k, & 1 \leq k \leq s \\ x_i, & k = s + i, \ i = 1, \ldots, m \end{cases} \quad \text{for } (x, y) \in Q_E$$

where the y_k and x_i are the kth and i^{th} components of y and x. A Pareto-efficient (minimum) point for $g_1 (x, y), \ldots, g_{s+m} (x, y)$ is a point $(x^*, y^*) \in Q_E$ such that there is no other point $(x, y) \in Q_E$ for which

$$g_k (x, y) \leq g_k (x^*, y^*), \quad k = 1, \ldots, s + m \qquad (2.3)$$

with at least one strict inequality. Evidently, the Pareto-efficient points of Q_E are those of P_E, hence we can restrict attention to P_E.

Charnes and Cooper (1961) showed (for **general**, multiple goals functions $g_k(x,y)$) that (x^*,y^*) is Pareto-efficient if and only if (x^*,y^*) is an optimal solution to the reduced, single mathematical (goal) program

$$\min \sum_{k=1}^{s+m} g_k(x,y)$$

$$g_k(x,y) \leq g_k(x^*,y^*), \quad k = 1, \ldots, s + m \qquad (2.4)$$

$$(x, y) \in P_E$$

The constraint inequalities in (2.4) for a test point (x^*, y^*) may be written as

$$y \geq y^*, \ x \leq x^*,$$

which are the envelopment constraints of DEA for an observed input vector x*
and corresponding output vector y*.

3. Pareto-Efficient Facets of the Empirical Production Possibility Set

As shown by the locus of points a b c d in Figure 2, the Pareto-efficient empirical production frontier is segmented into facets of efficient observed input-output points. The rates of change of outputs with respect to inputs along these efficient facets have important economic and managerial implicaitons for trade-off analysis and resource allocation. Our development in Sections 4 and 5 shows how these rates of change (which will be different along different facets) can be derived from the observed input-output points that lie on the efficient facets.

In this section we provide a theoretical basis for determining the efficient facets and a practical method for identifying the observed points on a facet. A small numerical example illustrates what is involved.

Figure 2.
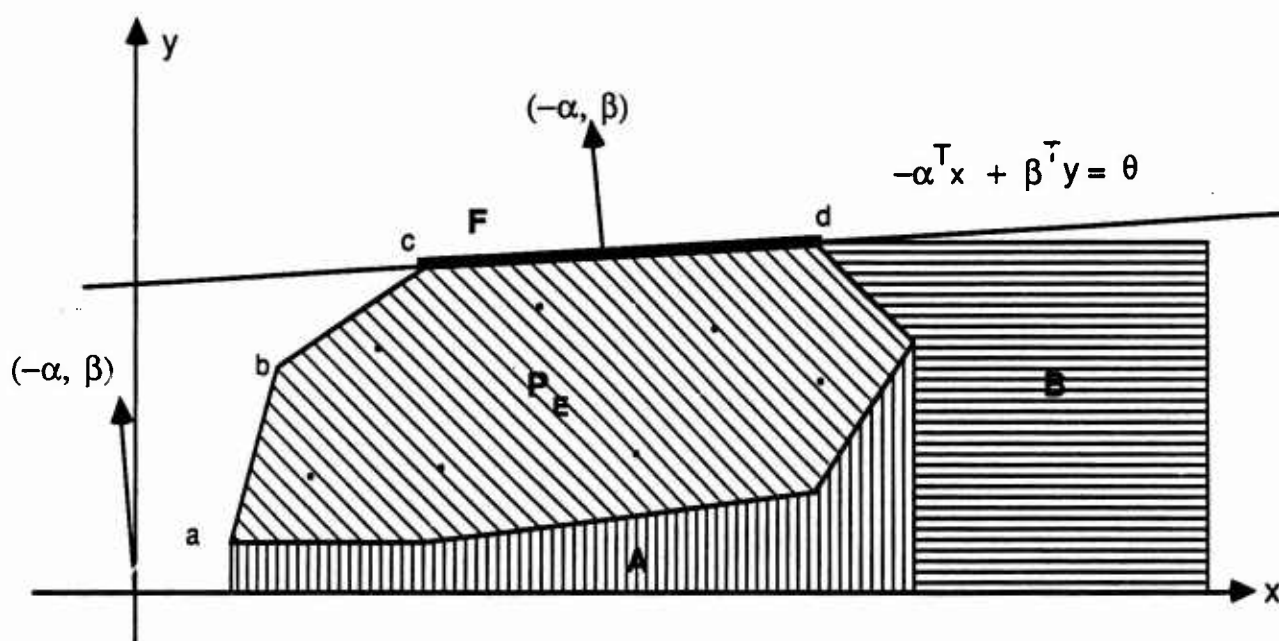
Let F be a facet of P_E contained in the hyperplane $-\alpha^T x + \beta^T y = \theta$ such that

$$-\alpha^T x + \beta^T y \le \theta \qquad \text{for all } (x,y) \in P_E \qquad (3.1)$$

as shown in Figure 2.

Theorem 3.1: Let the relative interior of F in the hyperplane $-\alpha^T x + \beta^T y = \theta$ be non-empty, and let at least one other observed point be outside F. Then F is an efficient facet if and only if $\alpha > 0$, $\beta > 0$.

Proof: By our assumption on F, int $(P_E) \ne \emptyset$.

($\Rightarrow$): Let $(\overline{x}, \overline{y})$ be a relative interior point of the facet F contained in $-\alpha^T x + \beta^T y = \theta$. Assume to the contrary that there exists $\alpha_i \le 0$ for some i $(i = 1, \ldots m)$ or $\beta_r \le 0$ for some r $(r = 1, \ldots, s)$.

(i) $\exists \, \alpha_i \le 0$ for some i.

Consider the new point $(x_\lambda, y_\lambda) = (\overline{x} - \lambda e_i, \overline{y})$ where e_i is a vector of zeros with a 1 in the ith position.

Then for any $\lambda > 0$ we have

$$-\alpha^T x_\lambda + \beta^T y_\lambda = -\alpha^T \overline{x} + \lambda \alpha_i + \beta^T \overline{y} \le -\alpha^T \overline{x} + \beta^T \overline{y} = \theta.$$

Hence, for small $\lambda > 0$ we have

$$(x_\lambda, y_\lambda) \in P_E$$

and

$$x_\lambda \le \overline{x}, \quad y_\lambda = \overline{y},$$

with strict inequality holding for x_i. This means that $(\overline{x}, \overline{y})$ is not efficient, which implies the facet F contained in $-\alpha^T x + \beta^T y = \theta$ is not efficient, a contradiction.

(ii) $\exists \, \beta_r \le 0$ for some r.

Consider the new point $(x_\lambda, y_\lambda) = (\overline{x}, \overline{y} + \lambda e_r)$ where e_r is a vector of zeros with a 1 in the rth position. Then for any $\lambda > 0$ we have

$$-\alpha^T x_\lambda + \beta^T y_\lambda = -\alpha^T \overline{x} + \beta^T \overline{y} + \lambda \beta_r \leq -\alpha^T \overline{x} + \beta^T \overline{y} = \theta.$$

Hence, for small $\lambda > 0$ we have

$$(x_\lambda, y_\lambda) \in P_E$$

and

$$x_\lambda = \overline{x}, \qquad y_\lambda \geq \overline{y}$$

with strict inequality holding for y_r. This means that $(\overline{x}, \overline{y})$ is not efficient, which implies the facet F contained in $-\alpha^T x + \beta^T y = \theta$ is not efficient, a contradiction

($\Leftarrow$) : Assume to the contrary that the facet F contained in $-\alpha^T x + \beta^T y = \theta$ with $\alpha > 0$ and $\beta > 0$ is not an efficient facet. That is, for some given relative interior point $(\overline{x}, \overline{y})$ of F, there exists another point $(\widehat{x}, \widehat{y}) \in P_E$ such that

$$\widehat{x} \leq \overline{x} \quad \text{and} \quad \widehat{y} \geq \overline{y}$$

with at least one strict inequality holding. But this implies

$$-\alpha^T \widehat{x} + \beta^T \widehat{y} > -\alpha^T \overline{x} + \beta^T \overline{y} = \theta,$$
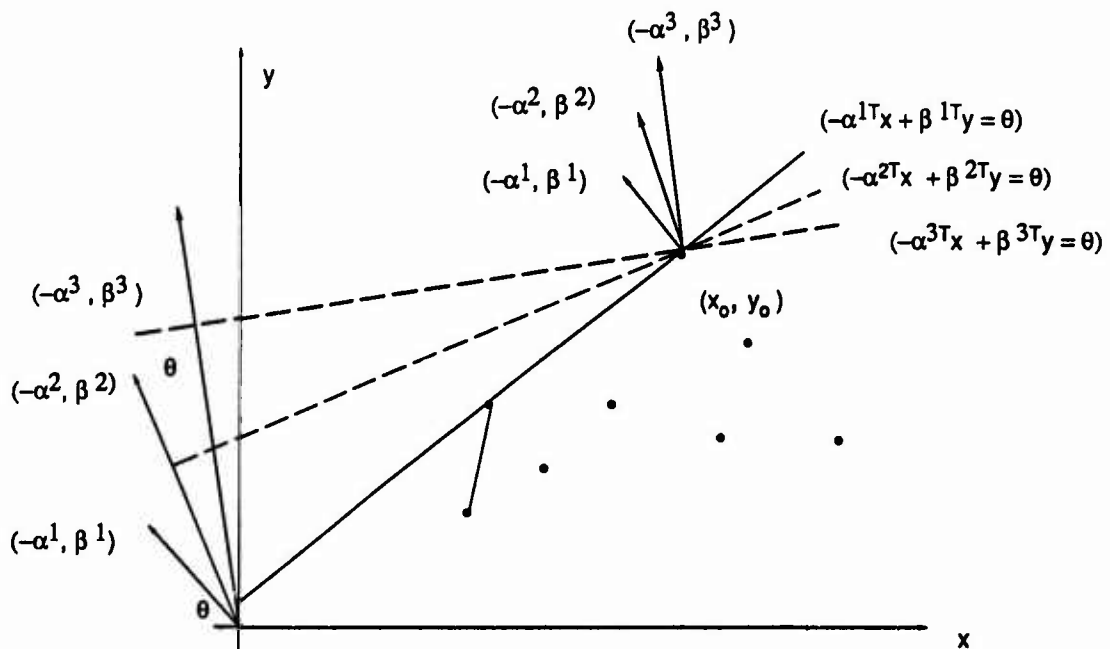
which contradicts (3.1).

<div align="right">Q.E.D.</div>

Now suppose that the empirical "input-output" point (x_0, y_0) is an efficient point. Consider the following linear programming problem.

$$\min \theta$$

$$-\alpha^T x_0 + \beta^T y_0 = \theta$$

$$-\alpha^T x_j + \beta^T y_j \leq \theta, \quad j = 1, \ldots, n \qquad (3.2)$$

$$\alpha^T e + \beta^T e = 1$$

$$\alpha \geq 0, \quad \beta \geq 0$$

Figure 3

Lemma 3.1: Let (α, β, θ) be a feasible solution of (3.2), and let $-\alpha^{i^T} x + \beta^{i^T} y = \theta_i$ with $\alpha^{i^T} e + \beta^{i^T} e = 1$ and $\alpha^i \geq 0, \beta^i \geq 0, i = 1, \ldots, k$, be all the facet hyperplanes passing through the efficient point (x_0, y_0) over $P_E \cup A \cup B$ (see Figure 3). Then there exists $\{\lambda_i\}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$ such that

$$(\alpha, \beta, \theta) = \sum_{i=1}^{k} \lambda_i (\alpha^i, \beta^i, \theta_i,)$$

Proof: Let $A = \{(x,y) : -\alpha^{i^T}x + \beta^{i^T}y \leq \theta_i, \ i = 1, \ldots, K\}$. Since (α, β, θ) is a feasible solution of (3.2), the hyperplane $-\alpha^T x + \beta^T y = \theta$ is a support plnae at (x_0, y_0) such that

$$-\alpha^T x_0 + \beta^T y_0 = \theta \tag{3.3}$$

and

$$-\alpha^T x + \beta^T y \leq \theta \quad \text{for all } (x, y) \in A \tag{3.4}$$

First we show that there exist $\{\lambda_i\}$ with $\lambda_i \geq 0$ such that $(\alpha, \beta, \theta) = \sum_{i=1}^{k} \lambda_i (\alpha^i, \beta^i \theta_i)$, that is,

$$(-\alpha, \beta, -\theta) = \sum_{i=1}^{k} \lambda_i (-\alpha^i, \beta^i -\theta_i),$$

Assume to the contrary that the following system (3.5) has no solution.

$$(-\alpha, \beta, -\theta) = \sum_{i=1}^{k} \lambda_i (-\alpha^i, \beta^i -\theta_i), \tag{3.5}$$

$$\lambda_i \geq 0, \ i = 1, \ldots, K.$$

By Färkas' theorem, there exist (d, γ, f) such that

$$-\alpha^{i^T}d + \beta^{i^T}\gamma - f\theta_i \leq 0, \ i = 1, \ldots k \tag{3.6}$$

and

$$-\alpha^T d + \beta^T \gamma - f\theta > 0 \tag{3.7}$$

By (3.7), (d, γ, f) must be a non-zero vector.

$$\text{(i) } f = 0$$

Consider $(\bar{x}, \bar{y}) = (x_0, y_0) + (d, \gamma)$. By (3.6) we have

$$-\alpha^{i^T}\bar{x} + \beta^{i^T}\bar{y} = (-\alpha^{i^T}x_o + \beta^{i^T}y_o) + (-\alpha^{i^T}d + \beta^{i^T}\gamma)$$

$$= \theta_i + (-\alpha^{i^T}d + \beta^{i^T}\gamma)$$

$$\leq \theta_i \quad \text{for all i}$$

That is, $(\bar{x}, \bar{y}) \in A$. But by (3.7) we have

$$-\alpha^T\bar{x} + \beta^T\bar{y} = (-\alpha^T x_o + \beta^T y_o) + (-\alpha^T d + \beta^T \gamma)$$

$$= \theta + (-\alpha^T d + \beta^T \gamma)$$

$$> \theta$$

This contradicts (3.4).

(ii) $f > 0$

Let $(\bar{x}, \bar{y}) = (\frac{d}{f}, \frac{\gamma}{f})$. Then by (3.5) we have

$$-\alpha^{i^T}\bar{x} + \beta^{i^T}\bar{y} = (-\alpha^{i^T}d + \beta^{i^T}\gamma)/f \leq \theta_i \quad \text{for all i.}$$

That is, $(\bar{x}, \bar{y}) \in A$. But by (3.7) we have

$$-\alpha^T\bar{x} + \beta^T\bar{y} = (-\alpha^T d + \beta^T \gamma)/f > \theta$$

This contradicts (3.4).

(iii) $f < 0$

Let $(\bar{x}, \bar{y}) = (x_0, y_0) + (x_0 - \frac{d}{f}, y_0 - \frac{\gamma}{f})$. Then by (3.6) we have

$$-\alpha^{i^T} \bar{x} + \beta^{i^T} \bar{y} = 2 (-\alpha^{i^T} x_0 + \beta^{i^T} y_0) + (-\alpha^{i^T} d + \beta^{i^T} \gamma)/(-f)$$

$$\leq 2 \theta_i - \theta_i = \theta_i \qquad \text{for all i}$$

That is, $(\bar{x}, \bar{y}) \in A$. But by (3.7) we have

$$-\alpha^T \bar{x} + \beta^T \bar{y} = 2 (-\alpha^T x_0 + \beta^T y_0) + (-\alpha^T d + \beta^T \gamma)/(-f)$$

$$> 2\, \theta - \theta$$

$$= \theta$$

This contradicts (3.4).

Hence there exist $\{\lambda_i\}$ with $\lambda_i \geq 0$ such that

$$(\alpha, \beta, \theta) = \sum_{i=1}^{k} \lambda_i (\alpha^i, \beta^i, \theta_i).$$

Since $\alpha^T e + \beta^T e = 1$ and $\alpha^{i^T} e + \beta^{i^T} e = 1$, i=1, ..., k,

we have

$$1 = \alpha^T e + \beta^T e = \sum_{i=1}^{k} \lambda_i \alpha^{i^T} e + \sum_{i=1}^{k} \lambda_i \beta^{i^T} e$$

$$= \sum_{i=1}^{k} \lambda_i (\alpha^{i^T} e + \beta^{i^T} e) = \sum_{i=1}^{k} \lambda_i$$

Q.E.D.

Theorem 3.2: Let $(\alpha^*, \beta^*, \theta^*)$ with $\alpha^* > 0$ and $\beta^* > 0$ be an optimal basic solution of (3.2) (e.g., using an extreme point method, such as the simplex method). Then $-\alpha^{*T}x + \beta^{*T} = \theta^*$ is a hyperplane containing one efficient facet passing through the efficient point (x_0, y_0) of P_E.

Proof: By Lemma 3.1 the feasible region of (3.1) is the convex hull of the $\{ (\alpha^i, \beta^i, \theta_i) \}$, where $-\alpha^{i^T}x + \beta^{i^T}y = \theta_i$, $i = 1, \ldots, k$ are all the hyperplanes containing all facets passing through (x_0, y_0).

When using an exteme point method, the optimal basic solution to (3.2) must be one of the $\{ (\alpha^i, \beta^i, \theta_i) \}$, say $\alpha^* = \alpha^1 > 0$, $\beta^* = \beta^1 > 0$, and $\theta^* = \theta_1$. By

Theorem 3.1: $-\alpha^{*T}x + \beta^{*T} = \theta^*$ is a hyperplane containing one efficient facet passing through (x_0, y_0).

$$Q.E.D.$$

Theorem 3.2 provides only a sufficient condition for an efficient facet. Since we cannot guarantee $\alpha^* > 0$ and $\beta^* > 0$, in practice we may use small numbers $\epsilon > 0$ and employ the following linear programming problem to derive the hyperplane containing one efficient facet passing through the efficient point (x_0, y_0) of P_E.

$$\text{Min } \theta$$

$$-\alpha^T x_0 + \beta^T y_0 = \theta \tag{3.8}$$

$$-\alpha^T x_j + \beta^T y_j \leq \theta, j = 1, \ldots, n$$

$$\alpha^T e + \beta^T e = 1$$

$$\alpha \geq \epsilon e, \beta \geq \epsilon e.$$

At an optimal solution $(\alpha^*, \beta^*, \theta^*)$ to (3.8), all those observed (efficient) points j which satisfy their respective constraints as equalities also lie on the efficient facet contained in the hyperplane passing throught (x_0, y_0). Such points, together with (x_0, y_0), constitute a subset (but not necessarily all) of the facet members. Applying (3.8) to other members of the facet will generally reveal additional points, and there will be duplication, overlapping and "resting" of these various subsets from which the facet may be identified by reduction. Thus, by applying (3.8) to each DEA-efficient point in turn, all efficient facets and their member points can be identified. This procedure requires little additional

computational effort, since moving from one efficient opint to the next involves changing only the first constraint of (3.8) with everything else unchanged.

To illustrate the above procedures consider the following two-output, two-input, 10-DMU example with data as given in Table 1.

Table 1.

DMU	Outputs		Inputs	
	y_1	y_2	x_1	x_2
1	2	1	9	9
2	3	1	12	8
3	2	2	7	12
4	5	3	6	10
5	4	4	10	5
6	3	3	8	10
7	6	6	12	10
8	8	2	14	6
9	1	6	12	12
10	3	5	8	8

The Pareto-efficient points are DMUs 4, 5, 7, 8 and 10. Applying program (3.8) to each efficient point in turn produces the results given in Table 2. By reduction we see that there are two efficient facets: (DMU 5, DMU 8) and (DMU 4, DMU 5, DMU 7, DMU 10). The facets, for both the outputs and the inputs, are depicted in Figure 4.
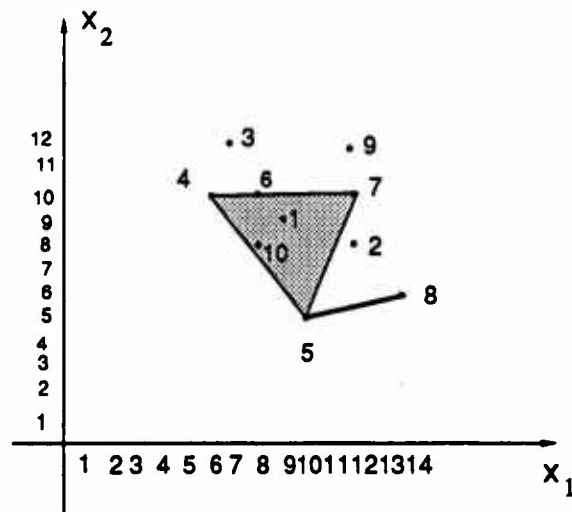
Table 2

Program (3.8) Applied to DMU	DMUs Revealed as Being in the Facet
4	4,5
5	4,5
7	4,5,7,10
8	5,8
10	4,5,10

Figure 4.
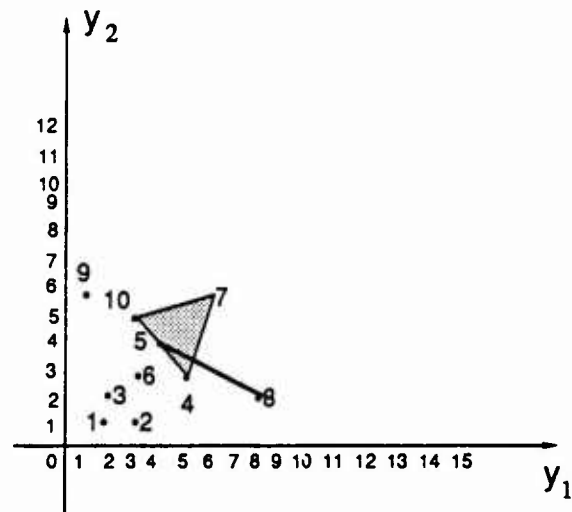
output side

input side

4. Determining Rates of Change

We begin this section with some definitions and lemmas important in our subsequent development.

Definition 4.1 Let S be an arbitrary set in E^m. The affine manifold spanned by S is given by $M(S) = \{x: x = \sum_{i=1}^{q} \lambda_i x^i, \ x^i \in S, \lambda_i \in E^1, \sum_{i=1}^{q} \lambda_i = 1, \ q \geq 1\}$

Definition 4.2 Let S be an arbitrary set in E^m. The linear subspace spanned by S is given by $L(S) = \{x: x = \sum_{i=1}^{q} \lambda_i x^i, x^i \in S, \lambda_i \in E^1, q \geq 1\}$

Lemma 4.1 Let S be an arbitrary set in E^m. Then for any $x \in S$ we have

$$M(S) = x + L(S-x)$$

Figure 5
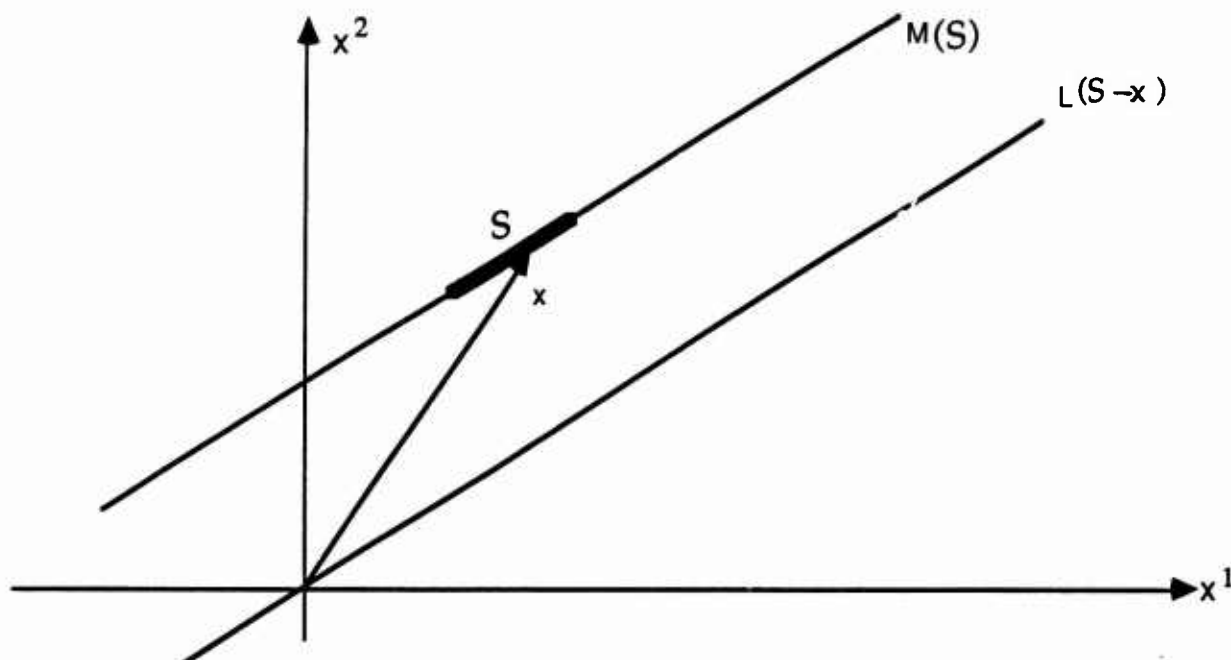
Definition 4.3 Let S be an arbitrary set in E^m. The convex hull of S is given by

$$K(S) = \{x: x = \sum_{i=1}^{q} \lambda_i x^i, \ x^i \in S, \lambda_i \geq 0, \sum_{i=1}^{q} \lambda_i = 1, \ q \geq 1\}$$

Lemma 4.2 Let $S = \{x^1, \ldots, x^k\}$ and Rank $(x^2-x^1, \ldots, x^k-x^1) = p-1$, and w. l. o. g. let $x^2-x^1, \ldots, x^p-x^1$ be linearly independent. Then

$$M(x^1, \ldots, x^p) = M(x^1, \ldots, x^k)$$

Proof : Since $L(x^2-x^1, \ldots, x^k-x^1) = L(x^2-x^1, \ldots, x^p-x^1)$, then by Lemma 4.1 we have

$$
\begin{aligned}
M(x^1, \ldots, x^k) &= x^1 + L(x^2-x^1, \ldots, x^k-x^1) \\
&= x^1 + L(x^2-x^1, \ldots, x^p-x^1) \\
&= M(x^1, \ldots x^p)
\end{aligned}
$$

Q.E.D.

Lemma 4.3 Let $\bar{x} = \frac{1}{p} \sum_{i=1}^{p} x^i$. Then $\bar{x}$ is a relative interior point of $K(x^1, \ldots x^k)$ in $M(x^1, \ldots x^k)$.

Proof: $\bar{x}$ is a relative interior point of $K(x^1, \ldots x^p)$ relative to $M(x^1, \ldots x^p)$. Since $M(x^1, \ldots x^p) = M(x^1, \ldots x^k)$ and

$K(x^1, \ldots x^p) \subset K(x^1, \ldots x^k)$, we have that $\bar{x}$ is a relative interior point of $K(x^1, \ldots x^k)$ in $M(x^1, \ldots x^k)$.

Q.E.D.

W.l.o.g., let $(x^1, y^1), \ldots, (x^k, y^k)$ be all the generators of a given efficient facet. Let Rank $(x^2-x^1, \ldots, x^k-x^1) =$ Rank $(x^2-x^1, \ldots, x^p-x^1) = p-1$, and let $\bar{x} = \frac{1}{p} \sum_{i=1}^{p} x^i$. By Lemma 4.3, $\bar{x}$ is a relative interior point of $(x^1, \ldots x^k)$. Hence, if we can determine the rates of change of the outputs with changes in the inputs at $\bar{x}$, we also have them at any point of that facet.

Let $\bar{x} - x^1 = \frac{1}{p} \sum_{i=2}^{p} (x^i - x^1) = X \, \hat{P}$

and

$$\bar{y} - y^1 = \frac{1}{p} \sum_{i=2}^{p} (y^i - y^1) = Y \, \hat{P}$$

where

$$X = (x^2 - x^1, \ldots, x^p - x^1)$$

$$Y = (y^2 - y^1, \ldots, y^p - y^1)$$

and $\qquad \hat{P} = (\frac{1}{p}, \ldots, \frac{1}{p})^T.$

Then

$$X^T (\bar{x} - x^1) = X^T X \, \hat{P}$$

$$\Rightarrow (X^T X)^{-1} X^T (\bar{x} - x^1) = \hat{P}$$

$$\Rightarrow \bar{y} - y^1 = Y (X^T X)^{-1} X^T (\bar{x} - x^1)$$

$$= W (\bar{x} - x^1)$$

where

$$W = Y (X^T X)^{-1} X^T.$$

For any r, we then have $\bar{y}_r - y_r^1 = {}_rW\,(\bar{x} - x^1)$ (4.1)

where ${}_rW$ denotes the rth row of W.

Now allow a smail change Δ_{i_o} in the particular input i_o , with all other input values unchanged.

Let

$$\bar{x}_{i_o} + \Delta_{i_o} - x_{i_o}^1 = \sum_{j=2}^{p} \alpha_j\,(x_{i_o}^j - x_{i_o}^1)$$

$$\bar{x}_i - x_i^1 = \sum_{j=2}^{p} \alpha_j\,(x_i^j - x_i^1),\ \ i \neq i_o$$

and

$$x\,(\Delta_{i_o}) = (\bar{x}_i^1, \dots, \bar{x}_{i_o}^1 + \Delta_{i_o}, \dots, \bar{x}_m)^T.$$

Then we have

$$x\,(\Delta_{i_o}) - x^1 = \sum_{j=2}^{p} \alpha_j\,(x^j - x^1) = X\,\alpha$$

and

$$y\,(\Delta_{i_o}) - y^1 = \sum_{j=2}^{p} \alpha_j\,(y^j - y^1) = Y\,\alpha$$

where

$$\alpha = (\alpha_2, \dots, \alpha_p)^T.$$

Then

$$\alpha = (X^T X)^{-1}\ X^T\ (x\,(\Delta_{i_o}) - x^1)$$

and

$$y\,(\Delta_{i_o}) - y^1 = Y(X^T X)^{-1}\ X^T\ (x\,(\Delta_{i_o}) - x^1)$$

$$= W\,(x\,(\Delta_{i_o}) - x^1).$$

Now,

$$\frac{y_r\,(\Delta_{i_o}) - \overline{y}_r}{x_{i_o}(\Delta_{i_o}) - \overline{x}_{i_o}} = \frac{(y_r\,(\Delta_{i_o}) - y_r^1) - (\overline{y}_r - y_r^1)}{\Delta_{i_o}}$$

$$= \frac{{}_r W\,(x\,(\Delta_{i_o}) - x^1) - (\overline{y}_r - y_r^1)}{\Delta_{i_o}}$$

$$= \frac{{}_r W\,(\overline{x} - x^1) + w_{r\,i_o}\Delta_{i_o} - (\overline{y}_r - y_r^1)}{\Delta_{i_o}}$$

$$= W_{r\,i_o}\quad (\text{by }(4.1)\,).$$

Thus, we have

$$\frac{\partial y_r}{\partial x_{i\,o}} = \lim_{\Delta_{i_o} \to 0} \frac{y_r\,(\Delta_{i_o}) - \overline{y}_r}{x_{i_o}\,(\Delta_{i_o}) - \overline{x}_{i_o}} = W_{r\,i_o} \qquad (4.2)$$

If Rank $(x^2-x^1, \ldots, x^k - x^1) < m$, then for some i_0 $\bar{x} + \Delta e_{i_0}$ will no longer lie on the facet for sufficiently small $\Delta > 0$. Therefore, we need to project e_{i_0} to the subspace $L(x^2-x^1, \ldots, x^p-x^1)$ and then obtain the rate of change along this projected direction. (See Figure 5)

Let $P = X(X^TX)^{-1}X^T$, so that P is a projection operator from the space E^m to the subspace $L(x^2-x^1, \ldots, x^p-x^1)$.

Let

$$d^{i_0} = \frac{Pe_{i_0}}{\|Pe_{i_0}\|},$$ where e_{i_0} has a 1 in the i_0^{th} position and zeros elsewhere. Then we have

$$\left. \frac{dy_r(x+\rho d^{i_0})}{d\rho} \right|_{\rho=0} = rWd^{i_0}, \, r = 1, \ldots, s; \, i_0 = 1, \ldots, m \tag{4.3}$$

Note that if Rank $(x^2-x^1, \ldots, x^k-x^1) = m$, then P is the $(m \times m)$ identity matrix.

Continuing with our illustrative example of the previous section, recall the two efficient facets were (5, 8) and (4, 5, 7, 10). We shall now determine the rates of change for these facets.

Efficient Facet (5, 8):

$$x^5 = \begin{pmatrix} 10 \\ 5 \end{pmatrix}, x^8 = \begin{pmatrix} 14 \\ 6 \end{pmatrix}, y^5 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, y^8 = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \text{ so that}$$

$x^8 - x^5 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $y^8 - y^5 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$. Since $x^8 - x^5 \neq 0$ it is linearly independent.

Figure 6.
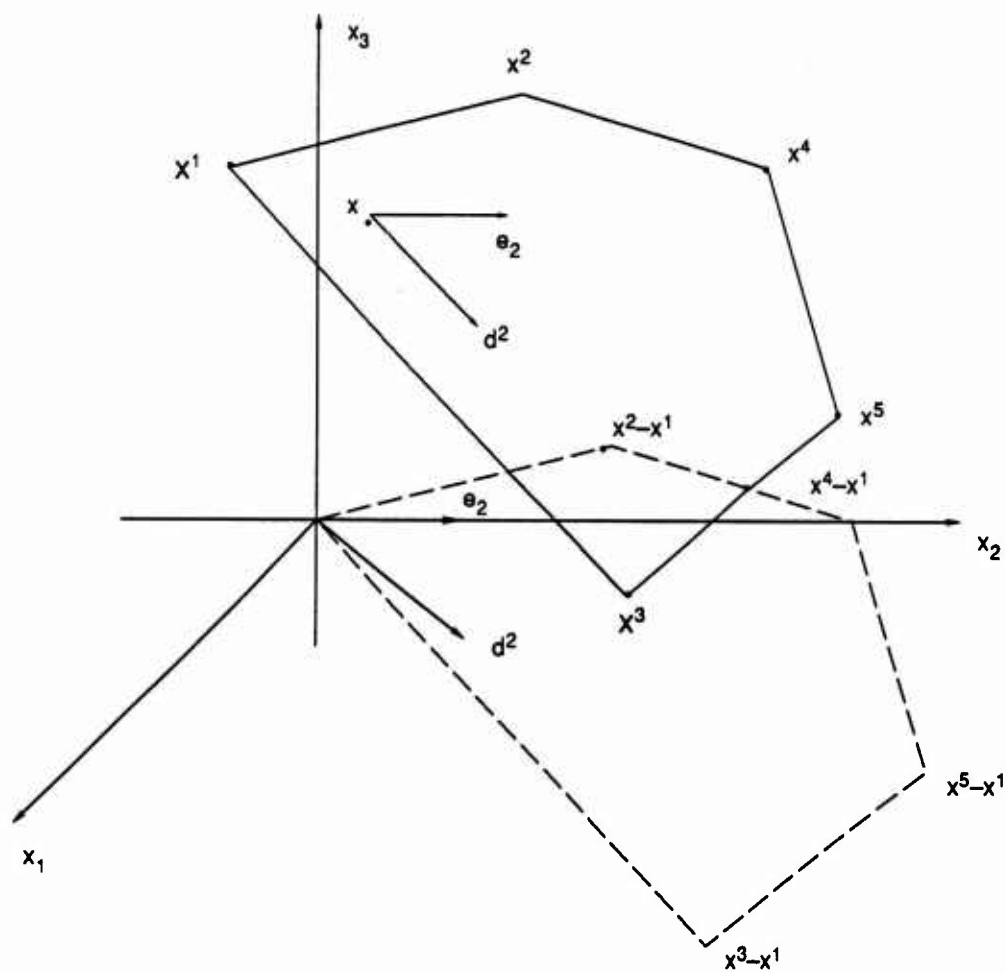
Thus

$$X = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \text{ and } (X^T X)^{-1} = \frac{1}{17}.$$

Therefore

$$W = Y \ (X^T X)^{-1} X^T = \frac{2}{17} \begin{pmatrix} 8 & 2 \\ -4 & -1 \end{pmatrix}$$

$$P = X \ (X^T X)^{-1} X^T = \frac{1}{17} \begin{pmatrix} 16 & 4 \\ 4 & 1 \end{pmatrix}$$

$$d^1 = \frac{Pe_1}{\| Pe_1 \|} = \frac{2}{\sqrt{272}} \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

$$d^2 = \frac{Pe_2}{\| Pe_2 \|} = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The rates of change are given by

$$\left. \frac{dy_1 \ (\bar{x}+\rho d^1)}{d\rho} \right|_{\rho = 0} = {}_1 W d^1 = \frac{\sqrt{272}}{17} \approx 0.97$$

$$\left. \frac{dy_1 \ (\bar{x}+\rho d^2)}{d\rho} \right|_{\rho = 0} = {}_1 W d^2 = \frac{4}{\sqrt{17}} \approx 0.97$$

$$\left. \frac{dy_2 \ (\bar{x}+\rho d^1)}{d\rho} \right|_{\rho = 0} = {}_2 W d^1 = -\frac{8}{\sqrt{272}} \approx -0.49$$

$$\left. \frac{dy_2 \ (\bar{x}+\rho d^2)}{d\rho} \right|_{\rho = 0} = {}_2 W d^2 = -\frac{2}{\sqrt{17}} \approx -0.49$$

Note that the last two rates of change are negative; we will address this situation in Section 5.

Efficient Facet (4, 5, 7, 10):

$$x^4 = \begin{pmatrix} 6 \\ 10 \end{pmatrix}, x^5 = \begin{pmatrix} 10 \\ 5 \end{pmatrix}, x^7 = \begin{pmatrix} 12 \\ 10 \end{pmatrix}, x^{10} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$$

$$y^4 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, y^5 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, y^7 = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, y^{10} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \text{ so that}$$

$$x^5 - x^4 = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, x^7 - x^4 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, x^{10} - x^4 = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \text{ and}$$

$$y^5 - y^4 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, y^7 - y^4 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, y^{10} - y^4 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

It is easily verified that $x^5 - x^4$ and $x^7 - x^4$ are linearly independent.

Hence

$$X = \begin{pmatrix} 4 & 5 \\ -5 & 0 \end{pmatrix}, Y = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$$

and

$$(X^T X)^{-1} = \begin{pmatrix} \dfrac{1}{25}, & \dfrac{-2}{75} \\ \dfrac{-2}{75}, & \dfrac{41}{900} \end{pmatrix}$$

Therefore

$$W = Y(X^T X)^{-1} X^T = \begin{pmatrix} \dfrac{1}{6}, & \dfrac{1}{3} \\ \dfrac{1}{2}, & \dfrac{1}{5} \end{pmatrix}$$

and

$$P = X(X^T X)^{-1} X^T = I_{2 \times 2} \quad .$$

Thus the rates of change in this facet are given by

$$\frac{\partial y_1}{\partial x_1} = \frac{1}{6}, \frac{\partial y_1}{\partial x_1} = \frac{1}{3}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{1}{2}, \frac{\partial y_2}{\partial x_2} = \frac{1}{5}.$$

5. Cone Directions for Non-negative Rates of Change

The rate of change of output y_r with respect to input x_i, as given by (4.3), is not guaranteed to be non-negative. When negative rates of change are given by (4.3) we may employ the following development.

Note first that the "projections" of the outputs of $y^1, \ldots, y^k$ along the direction $h \geq 0$ are given by the single outputs $\hat{y}^1 = h^T y^1, \ldots, \hat{y}^k = h^T y^k$. Now consider the following two DEA problems (5.1) and (5.2). They have the same input data, but the outputs of (5.2) are the projections (along the direction h) of the outputs of (5.1), and (x^o, y^o) is one observed point in $\{(x^1, y^1), \ldots, (x^n, y^n)\}$.

$$\min -e^T s^+ - e^T s^-$$

$$\Sigma\, y^j \lambda_j - s^+ \quad = y^o$$

$$-\Sigma\, x^j \lambda_j - s^+ \quad = -x^o \qquad\qquad (5.1)$$

$$\Sigma\, \lambda_j \quad\quad\quad = 1$$

$$\lambda_j,\, s^+, s^- \quad \geq 0$$

$$\min -h^T s^+ - e^T s^-$$

$$\Sigma \, h^T y^j \, \lambda_j - h^T s^+ \quad = h^T y^o$$

$$-\Sigma \, x^j \, \lambda_j \; - s^- \quad = -x^o \qquad\qquad (5.2)$$

$$\Sigma \, \lambda_j \qquad\qquad = 1$$

$$\lambda_j, \, s^+, s^- \quad \geq 0$$

If $h > 0$ then from (5.1) and (5.2) we have that (x^o, y^o) is efficient if and only if $(x^o, h^T y^o)$ is efficient, that is, $(x^o, \hat{y}^o)$ is efficient.

Now let

$$Y_h = (\hat{y}^2 - \hat{y}^1, \ldots, \hat{y}^p - \hat{y}^1)$$

$$= (h^T (y^2 - y^1), \ldots, h^T (y^p - y^1)$$

$$= h^T (y^2 - y^1, \ldots, y^p - y^1)$$

$$= h^T Y$$

and

$$W_h = Y_h \, (X^T X)^{-1} \, X^T$$

$$= h^T Y \, (X^T X)^{-1} \, X^T$$

$$= h^T W$$

We then have

$$\frac{d\hat{y}\,(\bar{x}+\rho d^i)}{d\rho}\Bigg|_{\rho=0} = W_h d^i = h^T W d^i, \quad i = 1, \ldots, m \qquad (5.3)$$

Thus, if in (4.3) there exist some r and i such that $_r W d^i < 0$, we need to find a direction $h > 0$ such that

$$_r W d^i \geq 0, \quad i = 1, \ldots, m \qquad (5.4)$$

Case (i): If (5.4) has at least one positive solution, then it is in the cone of directions which contains nonnegative rates of change on this efficient facet.

Case (ii): (5.4) has no positive solution and Rank $(x^2-x^1, \ldots, x^k-x^1) < m$. The hyperplane $-\alpha^{*T} x + \beta^{*T} y = \theta^*$ which contains the efficient facet has been determined by the linear programming problem (3.8). Now set some suitable input vector x^* such that $x^2-x^1, \ldots, x^p-x^1, x^*-x^1$ are linearly independent, and determine the corresponding efficient output vector y^* by solving $-\alpha^{*T} x^* + \beta^{*T} y^* = \theta^*$. We thus have a new efficient point (x^*, y^*) and an extended efficient facet. The corresponding new X^*, Y^*, W^* and P^* are given by

$$X^* = (x^2-x^1, \ldots, x^p-x^1, x^*-x^1)$$

$$Y^* = (y^2-y^1, \ldots, y^p-y^1, y^*-y^1)$$

$$W^* = Y^* (X^{*T} X^*)^{-1} X^{*T}$$

$$P^* = X^* (X^{*T} X^*)^{-1} X^{*T}$$

and

$$\overline{d}^i = \frac{P^* e_i}{\| P^* e_i \|}, \text{ where } e_i \text{ is a vector of zeros with a 1 in the } i^{th} \text{ position.}$$

Then by (4.3) we have

$$\frac{dy_r (\overline{x}+\rho d^i)}{d\rho} \bigg|_{\rho=0} = {}_r W^* \overline{d}^i, \quad r = 1, \ldots, s; \, i = 1, \ldots, m \qquad (5.5)$$

If the rates of change given by (5.5) are still negative, the above procedure is repeated, further extending the efficient facet one point at a time. The process stops after at most $m+1-p$ iterations.

Case (iii): (5.4) has no positive solution and Rank $(x^2-x^1, \ldots, x^k-x^1) = m$.

In this case we can determine directions in the inputs such that the rates of change of outputs with respect to inputs are nonnegative.

We need to find a direction d which lies on the subspace

$L (x^2-x^1, \ldots, x^k-x^1)$ such that

$$\frac{dy_r (\overline{x}+\rho d)}{d\rho} \bigg|_{\rho=0} = {}_r Wd \geq 0, \qquad \text{for all } r = 1, \ldots, s.$$

Since d lies on $L (x^2-x^1, \ldots, x^k-x^1)$, d has the representation

$$d = Pz, \qquad \text{for some } z \in E^m.$$

This means we must have

$${}_r WPz \geq 0, \qquad \text{for all } r = 1, \ldots, s$$

i.e.,

$$WPz \geq 0.$$

Then for any direction d in the cone Λ given by

$$\Lambda = \{d:\ d = Pz,\ WPz \geq 0\} \qquad (5.6)$$

we have the rate of change

$$\left.\frac{dy_r\,(\bar{x}+\rho d)}{d\rho}\right|_{\rho=0} = rWd \geq 0, \qquad \text{for all } r = 1, \ldots, s. \qquad (5.7)$$

i.e.,

$$Wd \geq 0 \qquad (5.8)$$

Recall that for the efficient facet (5, 8) of our illustrative example, the rate of change of the second outpuf with respect to each input was negative. To derive nonnegative rates of change along this facet we need to determine a direction $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} > 0$ such that

$$h^T Wd^i \geq 0, \quad i = 1, 2.$$

Thus we have

$$\left.\frac{d\hat{y}\,(\bar{x}+\rho d^1)}{d\rho}\right|_{\rho=0} = h^T Wd^1 = \frac{4}{17\sqrt{272}} \quad (68\,h_1 - 34\,h_2) \geq 0$$

$$\left.\frac{d\hat{y}(\bar{x}+\rho d^2)}{d\rho}\right|_{\rho=0} = h^T W d^2 = \frac{2}{17\sqrt{17}} \quad (34\,h_1 - 17\,h_2) \geq 0$$

which imply that such a direction must be in the cone

$$2\,h_1 \geq h_2.$$

We conclude this section by highlighting the distinction between the single-output and the multiple-output cases. Charnes et al (1985) showed that if a Pareto-efficient empirical production function has only a single output, then it is an isotone function. Hence, if Rank ($x^2-x^1, \ldots, x^k-x^1$) = m, we can always obtain nonnegative rates of change. If Rank ($x^2-x^1, \ldots, x^k-x^1$) < m, we can extend the facet by the procedure given for case (ii) above, and thus will be quaranteed nonnegative rates of change. In contrast, as has been shown in this paper, the multiple output case is considerably more complex.

6. Conclusion

The present paper has extended the existing theory of Data Envelopment Analysis to develop what rates of change of outputs with changes in inputs can be determined on the Pareto-efficient facets of an empirically defined production possibility set. These rates of change, which will be different on different facets, are important for effective management of the resources (inputs) employed to obtain desired feasible outputs.

The efficient facets can be obtained by solving a series of linear programming problems, one for each Pareto-efficient observed input-output point. It is shown that the rates of change can then be computed from any linearly independent (in the inputs) subset of the facet's points. For the single output case

we always obtain nonnegative rates of change. The multiple output case is more complex, and nonnegativity is not guaranteed. However, the "cone direction" development of Section 5, in the output space or in the input space, shows in what directions change must go to obtain nonnegative rates of change. A simple example was developed and carried throughout the paper to provide a clearer understanding of the geometry of the empirical Pareto-efficient functions as well as to clarify the steps in our procedures.

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Efficient Facets for the additive model of DEA (Data Envelopment Analysis) are obtained by a new series of linear programming models which are used to show how rates of change of outputs with respect to inputs, both analytically and computationally, may be determined along a given efficient facet. The differences between single and multiple output cases are determined and exemplified via a "cone direction" development.

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